Finally we obtain the upper bound for $P_{2/u1}$ and thus $P_2$ as

$$P_2 \leq \sum_{i=1}^{L-1} \int_{-\infty}^{\infty} \sum_{k=1}^{p_1} n_k \phi(z-a_{ik}) \prod_{j=1}^{p_i} \left( \Phi(z-a_{ij}) \right)^{n_j} dz,$$

$$\cdot \left[ 1 - \prod_{i=1}^{L-1} \left( 1 - \Phi(z-a_{ik,1,i}) \right)^{n_{i+1}} \right] dz$$

$$+ \int_{-\infty}^{\infty} \sum_{i=1}^{p_i} n_i \left( \Phi(z-a_{ii}) + \phi(z+a_{ii}) \right) \left( \Phi(z-a_{ii}) - \Phi(-z-a_{ii}) \right) dz$$

$$\cdot \prod_{j=1}^{p_i} \left( \Phi(z-a_{ij}) \right)^{n_j} \left( \Phi(z-a_{ij}) - \Phi(-z-a_{ij}) \right)^{n_j}.$$

### References


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**Group Random-Access Disciplines for Multi-Access Broadcast Channels**

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*Abstract—* A group random-access (GRA) control discipline for a multi-access communication channel is presented and studied. A GRA scheme uses only certain channel time periods to allow some network terminals to transmit their information-bearing packets on a random-access basis. The channel can thus be utilized at other times to grant access to other terminals or other message types, by applying as appropriate group random-access, reservation, or fixed access-control procedures. GRA schemes could also be utilized to provide channel access to various network protocol packets. The average packet delay under a GRA discipline is evaluated by a Markov ratio limit theorem. To stabilize the channel, the GRA procedure is controlled dynamically by a control policy that rejects any newly arriving packets within certain time periods. Studying the associated Markov decision problem, the optimal control policy is characterized as yielding a minimal average packet delay under a prescribed maximal packet probability of rejection. This policy is shown to be represented by a single-threshold scheme. For such a scheme, a threshold value that attains the minimal probability of rejection is shown to exist and to yield a desirable control procedure. Performance curves are presented to demonstrate the delay-throughput characteristics induced by GRA procedures.

I. INTRODUCTION AND SYSTEM DESCRIPTION

We consider a multi-access broadcast channel serving a network of terminals. A satellite communication channel, a radio channel in a terrestrial radio network, or a communication cable link in a computer communication network serve as a few examples. The channel utilizes a repeater (such as a satellite transponder or a radio relay station) to enable each terminal in the network to communicate through the repeater with any other terminal [1]. Messages transmitted by the terminals are directed through the channel uplink to the repeater. The latter then shifts the message uplink frequency band into a disjoint downlink frequency-band and broadcasts the messages (so that each terminal can receive any signal reflected by the transmitter) through the downlink channel towards the network terminals. Note that no schedulings for message downlink transmissions are required.

We assume a synchronized structure. Thus time (referenced with respect to repeater's time) is divided into fixed-length durations of $\tau$ s, called slots. An appropriate network synchronization procedure is used to achieve network slot synchronization. Terminals will start transmissions of messages only at times coinciding with starting times of the synchronized time slots. The channel is characterized by a propagation delay of $R \tau$ s or $R$ slots. Propagation delays are of the order of milliseconds for packet radio channels and about $0.25$ s for satellite channels. Each terminal message is considered to be described as a packet of fixed length of $\mu^{-1}$ bits (including protocol, information, and parity-check bits). Information is transmitted through the channel at a rate of $C$ bits/s. Packet transmission time across the channel is thus $(\mu C)^{-1}$ s. We
We consider a network of $\tilde{M}$ terminals. New messages arrive at the $i$th terminal, $i=1,2,\cdots, \tilde{M}$, according to a Poisson stream of intensity $\lambda_i$. The overall network message (or packet) arrival stream is thus a Poisson point process with intensity

$$\lambda = \sum_{i=1}^{\tilde{M}} \lambda_i \text{ message/slot}. $$

Upon the arrival of a new message, a terminal will immediately try to gain access into the channel for this message. No terminal-buffer capacity or blocking constraints are imposed. One can also consider a terminal to possess buffer storage for only a single message and subsequently to be blocked for new arrivals when occupied. However, when the network contains a large number of active low duty cycle bursty terminals, the blocking effects would be insignificant.

To utilize efficiently the bandwidth of such a channel and grant acceptable message response times to the terminals sharing this channel, one needs to apply an appropriate access-control discipline. Various access-control procedures employing reservation schemes have been recently studied (see [1]–[2] and references therein). Using these schemes, each terminal needs to transmit a reservation packet to reserve a slot (or number of slots) for a newly arrived message. Assuming a decentralized control mechanism, each terminal (while receiving the broadcasted reservation packets) stores in its own queuing table the present state of the reservation process, being subsequently able to determine its own allocation of transmission slots. In a centralized control mode, a central controller receives all the reservation packets and subsequently instructs the terminals when to transmit their messages. Dynamic reservation schemes, considering single and multipacket messages, are shown in [1] to yield excellent delay-throughput performance characteristics over the whole range of moderate to high network traffic intensity values.

For low network traffic intensity values, when single-packet highly bursty terminal message processes are considered, a better delay-throughput performance involving a much less sophisticated (distributed) access-control procedure can be achieved by a random-access mechanism. The latter allows terminals to use the channel at any time to transmit a newly arrived packet. If, however, two or more packets collide, the involved messages are retransmitted after a random time period. Packet collisions may then occur if two or more packets are transmitted in the same slot. Colliding packets are retransmitted after a random time period. Clearly when uncontrolled, the number of colliding packets will eventually become unbounded with probability one. Therefore the SA channel needs to incorporate a flow control mechanism to avoid instabilities.

In many cases it is desirable to dedicate only certain portions of the channel time frame to a family of network terminals wishing to share the channel (during these periods) on a random-access basis. For that purpose, we present the group random access (GRA) discipline. Under a GRA discipline, a group of network terminals is provided with a periodic sequence of channel-access periods, during which this group uses a random-access discipline to gain access into the channel. A packet experiencing collision during a certain period will be retransmitted during the next access period. Other groups of network terminals (distinguished by their priorities, performance requirements, or by the statistics and nature of their information emitting, for example, short-interactive or longer long-haul messages) can share the remaining time-frame duration using again GRA procedures or other access-control techniques.

It is many times of particular interest to use a GRA procedure to grant channel access to certain protocol packets. The latter packets are usually much shorter than the message packets so that low throughput values are acceptable. At the same time, the simple distributed-control structure of the GRA is highly desirable. This is the case when reservation access-control disciplines are considered and (shorter) reservation packets need to be transmitted by the terminals [1], [2]. The latter reservation packets can be assigned periodical reservation periods during which they use a random-access procedure to compete for channel access. This procedure clearly results in a GRA access-control mechanism, utilized by the family of reservation packets.

The approximate throughput and delay-throughput performance of a regular (slotted ALOHA) random-access procedure has been studied (see [3]–[8] and references therein), assuming an approximating Markovian channel state process. Certain dynamic control schemes that stabilize the inherently unstable slotted ALOHA channel have also been investigated (by proposing certain threshold
control schemes, not necessarily optimal, and computing their performance through the associated dynamic-programming equation, [6], [8].

In this paper we present a precise study of the performance of a group random-access discipline and its optimal dynamic control. The channel is controlled so that the minimal average message delay is attained under an appropriately prescribed packet probability of rejection. The GRA procedure is shown to yield delay-throughput performance characteristics comparable to those attained by a regular (slotted ALOHA) random-access procedure while providing the network designer with a much higher degree of flexibility in granting access to different classes of information and protocol messages.

We note that our study of the GRA procedure as an access-control discipline for a multi-access communication channel can be readily applied also to nonbroadcast channels where a central controller (or other means) is incorporated to provide the terminals with the relevant (positive or negative) acknowledgment and control information.

The GRA procedure is presented in Section II where we also derive the characteristics of its underlying Markov state sequence and obtain a formula for the limiting average packet delay. We note that as for an SA procedure, the GRA scheme yields a channel throughput not larger than \( e^{-1} \) [packets/slot] and requires the incorporation of a flow control mechanism to avoid unbounded limiting packet delays. An optimal dynamic control policy for a GRA channel is characterized and studied in Section III. A class of binary control strategies that either accept or reject all new packet arrivals within each period is considered. It is also assumed that all colliding packets in one period must be retransmitted in the next period and that the variable representing the total number of colliding packets in a group can be observed. In many actual situations terminals can, however, only observe whether collisions have or have not occurred within a slot. It is nevertheless shown in Section III that a scheme using the latter observations will exhibit a delay-throughput performance similar to that obtained by a corresponding scheme using the former observations.

It is shown in Section III that the solution to an associated Markov decision problem induces the optimal GRA control function that yields the minimal average packet delay under a prescribed maximal value of probability of rejection. The optimal GRA control strategy is then shown to involve a single-threshold control scheme. The latter accepts or rejects all new packet arrivals in a period depending on whether the overall number of collisions in the previous period is, respectively, below or above the scheme threshold. Performance curves for an optimally controlled GRA channel are then presented. The appropriate preferable structure (threshold values) for a GRA channel controller is subsequently noted. The controlled GRA channel is observed to exhibit low packet delays over the whole range of acceptable network traffic intensities (or rejection probabilities).

II. THE GROUP RANDOM-ACCESS DISCIPLINE

Network Performance Measures

We consider a synchronized multi-access broadcast communication channel of capacity \( C \) bits/s, a slot duration of \( r \) s, and propagation delay of \( R \) slots. The channel serves a large community of \( M \) terminals generating new messages according to a Poisson stream of intensity \( \lambda \) message/slot. Each message is considered to be a packet of fixed length of \( \mu^{-1} \) bits. The packet transmission time is set to be equal to the slot duration \( \tau = (\mu C)^{-1} \).

The performance of an access-control discipline applied to this channel is assessed in terms of the following measures. A performance indicator of major importance is the steady-state average packet waiting-time function \( \bar{W} \). The latter can be expressed as the limit (when it exists)

\[
\bar{W} = \lim_{N \to \infty} N^{-1} E \left( \sum_{n=1}^{N} W_n \right) \tag{2.1}
\]

where \( W_n \) denotes the waiting-time of the \( n \)th message in the system. This waiting time, expressed in terms of number of slots, is measured from the instant the packet is transmitted by its terminal to the instant the packet is successfully transmitted. The overall steady-state average message delay \( D \) includes also single-slot and \( R \)-slot durations to account for the transmission time and propagation delay, respectively, associated with a successful packet transmission. We thus have

\[
D = \bar{W} + R + 1. \tag{2.2}
\]

A successful packet transmission will occur if the packet is the only one being transmitted in its slot, while packet collisions occur if more than a single packet is being transmitted in the same slot. Denoting by \( \bar{S}_i \), the number of successful transmissions in the \( i \)th slot, \( \bar{S}_i = 0, 1, i > 1 \), the channel throughput is given by

\[
s = \lim_{N \to \infty} N^{-1} E \left( \sum_{i=1}^{N} \bar{S}_i \right) \tag{2.3}
\]

expressing the channel output rate (i.e., the limiting average number of successful packets/slot).

Incorporating a control function to stabilize the GRA channel, certain packets will be denied access and be (at least temporarily) rejected. The probability \( P_R \) indicating the probability of packet rejection will thus serve as another index of performance. Note that if access (eventual successful transmission) is then provided to all nonrejected (accepted) packets, we will have

\[
s = \lambda (1 - P_R). \tag{2.4}
\]

The Access Control Procedure

The regular slotted ALOHA (SA) random-access discipline operates as follows.

Protocol (SA Discipline): A newly arrived packet is transmitted by its terminal at the start of the next slot. A packet transmission (or retransmission) that collides with
other packet transmissions is retransmitted by its terminal. The retransmission slot is chosen according to a uniform distribution over the $L$ slots following the reception of the broadcasted collision (i.e., $R$ slots after the transmission of the latter colliding packet). Each packet is retransmitted, governed by the latter random retransmission delay procedure, until it is successfully transmitted (avoiding any collisions).

For delay–throughput analysis of a broadcast channel under the SA scheme, the reader is referred to [3]–[8]. In particular, we note that channel throughput under a SA procedure is not larger than $e^{-1}$ [packets/slot]. Furthermore, the admission of packets into the system needs to be controlled to avoid unbounded (limiting) packet delays.

To present the group random-access (GRA) procedure, we identify first the sequence of periods $\{B_n, n > 1\}$, during which the group of terminals under consideration are allowed to contend for channel access. The $n$th period $B_n$ is assumed to contain $K$ successive channel slots $K > 1$ and thus be of duration $Kr$ s. The distance between $B_n$ and $B_{n+1}$, measured in the number of slots following $B_n$ and preceding $B_{n+1}$, can be taken to be given by any fixed number of slots not smaller than the propagation delay $R$. We assume in the following analysis the latter distance to be equal to $P$, $P > R$, for any $n$, $n > 1$.

If $N$ groups of terminals are set to utilize the whole channel capacity, each using a GRA procedure, channel time is decomposed into the union of $N$ periodic sequences $\{B_n, n > 1\}$, so that $B_n$ is followed by $B_{n+1}$, $i < N$, and $B_{n+1}$ by $B_{n+2}$, $n > 1$. The $i$th group of terminals with an overall traffic rate $\lambda_i$ uses the channel only during periods $\{B_n, n > 1\}$ and thus does not interfere with the transmissions of any other group of terminals. Since the performance characteristics are then the same for any group of terminals, except for possibly varying values of $\lambda$, $R$, $P$, and $K$, we need to study the performance of only a single group of terminals that we assume to have an overall traffic intensity $\lambda$ and to emit newly arriving packets following Poisson statistics only within the sequence of periods $\{B_n, n > 1\}$ and is thus not interfered with the transmissions of any other group of terminals. Since the performance characteristics are then the same for any group of terminals, except for possibly varying values of $\lambda$, $R$, $P$, and $K$, we need to study the performance of only a single group of terminals that we assume to have an overall traffic intensity $\lambda$ and to emit newly arriving packets following Poisson statistics only within the sequence of periods $\{B_n, n > 1\}$. Thus we let $A_n$ denote the number of new arrivals during the $n$th period $B_n$ and note that $\{A_n, n > 1\}$ is a sequence of i.i.d. random variables governed by a Poisson distribution

$$P(A_n = j) = (K\lambda)^j/j! \exp(-K\lambda), \quad j = 0, 1, 2, \ldots,$$

(2.5)

Note that we can alternatively assume new packets to arrive at any time, following Poisson statistics with rate $\lambda_0$ [packets/slot]. These packets are still served on a GRA basis and can therefore utilize only periods $\{B_n, n > 1\}$ for transmissions. In this case we can assume that all packets arriving during time interval $T_n$ defined as the time epoch including the $P$ slots that precede $B_n$, will be uniformly distributed for transmission within $B_n$. Subsequently, the sequence $\{A_{n+1}\}$ of new arrivals within $B_{n+1}$ is still an i.i.d. sequence of Poisson random variables governed by distribution (2.5) with $K\lambda$ replaced by $(P + K)\lambda_0$ or just $\lambda$ replaced by $(1 + P/K)\lambda_0$. We note that packets arriving outside a period $B_n$ will need to be stored in their terminal buffers prior to their transmission in a $B_n$ period. (Or just lock their terminal upon arrival, when terminals containing only single packet buffers are considered. Model (2.5) is then still appropriate for large enough $M$.) The latter storage times will result with an additional packet delay component expressing the packet waiting time from arrival until first transmission. This extra delay term is not included in the following analysis but can be appropriately added.

Considering henceforth the above-mentioned arrival patterns allowing new packets to arrive only within a $B_n$ period, the GRA scheme operates as follows.

**Protocol (GRA Discipline):**

1) Newly arrived packets are transmitted in the next slot if admitted by the network control procedure. Otherwise these packets are rejected.
2) Packets colliding within $B_n$ are retransmitted within $B_{n+1}$, $n > 1$, at a slot determined by a uniform distribution over $[1, K]$.
3) Each packet is transmitted and retransmitted until successfully transmitted or until rejected from the network by the network control procedure.

We note that steps 1) and 3) in the GRA protocol incorporate the possibility of packet rejection control to yield finite packet average delay times as noted for the SA scheme. The related optimal control analysis will be presented in the next section. We will present in this section a few basic characteristics of the GRA scheme.

We let $N_n$, $R_n$, and $S_n$ denote the numbers of transmissions, collisions, and successful transmissions, respectively, within the $n$th period $B_n$. The corresponding numbers within the $i$th slot of the $n$th period are denoted by $N_n(i)$, $R_n(i)$, and $S_n(i)$, respectively. We thus have

$$R_n = \sum_{i=1}^{K} N_n(i)I(N_n(i) > 2),$$

(2.6)

$$S_n = \sum_{i=1}^{K} I(N_n(i) = 1),$$

(2.7)

$$N_{n+1} = R_n + A_n + S_{n+1} + R_{n+1},$$

(2.8)

where $I(A)$ is the indicator function associated with event $A$, so that $I(A) = 1$ if $A$ occurs and $I(A) = 0$ otherwise.

The GRA channel state process can be represented as a vector Markov chain $Z = \{Z_n, n > 1\}$ over the space $\mathbb{N}_K$ where $\mathbb{N}$ denotes the set of nonnegative integers,

$$Z_n = \{T_n(i), A_n(i), i = 1, 2, \ldots, K\}$$

and $A_n(i)$ and $T_n(i)$ denote the number of new arrivals and retransmissions, respectively, allocated to the $i$th slot within $B_n$, $n > 1$, $1 \leq i \leq K$. The transition probability function of $Z_n$ is obtained as follows. We note that, in the absence of input control, the arrival variables $\{A_n(i)\}$ are statistically independent of $Z_n$ and are characterized by (2.5). The variables $\{T_n(i)\}$ depend on $Z_n$ only through...
Fig. 1. Transition $R_n \rightarrow R_{n+1}$ for Markov state chain $R$ associated with GRA discipline.

$R_n$, which is given by (2.6) as

$$R_n = \sum_{j=1}^{K} R_n^{(j)},$$

(2.9a)

where for each $j, j=1,2; \cdots, K$,

$$R_n^{(j)} = N_n^{(j)} I(N_n^{(j)} > 2)$$

(2.9b)

and

$$N_n^{(j)} = T_n^{(j)} + A_n^{(j)},$$

(2.9c)

for $n > 1$. Since the position of a retransmission is determined by a uniform distribution over $[1, K]$ we find $(T_n^{(j)})$ to be governed by the multinomial distribution

$$P\left\{ T_n^{(j)} = n_j, 0 < n_j < j, 1 < i < K | R_n = j \right\} = g_i^j(n_1, \cdots, n_K)$$

(2.10)

where $0 < n_i < j, i=1,2; \cdots, K$, $\sum_{i=1}^{K} n_i = j$. Equations (2.5), (2.9), and (2.10) thus yield the transition probability function for the Markov state chain $Z$. One observes that $Z_{n+1}$ depends on $Z_n$ only through $R_n$. In particular, we note that $R = \{ R_n, n > 1 \}$ is a Markov chain over the space of nonnegative integers $\mathbb{N}$ with a transition probability function expressed through (2.9), (2.10). A flow diagram indicating the transition $R_n \rightarrow R_{n+1}$ is shown in Fig. 1.

The maximal throughput of a GRA channel is upper bounded as follows. The throughput variable $S_{n+1}$ is given by

$$S_{n+1} = \sum_{j=1}^{K} I(N_n^{(j)} = 1).$$

(2.11)

By the multinomial distribution (2.10) and distribution (2.5), we conclude that $N_n^{(j)}$ ((2.9c)) is governed by a binomial distribution given $N_n^{(j)} = A_n^{(j)} + R_n$, $1 < k < K$:

$$P\left\{ N_n^{(k)} = j | N_n^{(j)} = j \right\} = \binom{j}{K} \left( \frac{1}{K} \right)^{j \frac{1}{K} - 1}$$

(2.12)

where $0 < i < j, j > 0$. Hence by (2.11), (2.12), we obtain

$$E(S_{n+1}) = E \left[ N_n^{(j)} \left( 1 - \frac{1}{K} \right)^{N_n^{(j-1)}} \right].$$

(2.13)

The GRA channel throughput $s$, expressing the average number of successful transmissions/slot, is given by (2.3) with $S_n$ replaced by $K^{-1} S_n$. Noting that $xa^x < e^{-[\ln(a-1)]^{-1}}$, for $x > 0, a > 1$, and, using (2.3), (2.13), we obtain $s$ to be upper bounded as

$$s \leq \frac{1}{e} \left( \frac{1}{K-1} \right) \left( \frac{1 + (K-1)^{-1}}{1} \right) \approx \frac{1}{e} G_K.$$  

(2.14)

It is noted that the maximal value of $E \{ S_{n+1} | N_n^{(j)} \}$ is attained at $N_n^{(j)} = N_n^{(j)}$, where $N_n^{(j)}$ is given by the integer part of $\left( \frac{\ln (1 + (K-1)^{-1})^{-1}}{1} \right)$. Note also that $N_n^{(j)} > K-1$. The function $G_K$ is very close to 1 for any $K > 2$ and $G_K \rightarrow 1$ as $K \rightarrow \infty$. We note that actual GRA schemes (see Figs. 1-5) practically attain $s = (1/e)G_K$. We have thus proved the following result.

Theorem 1: The throughput of a GRA scheme with $K$ slot periods is upper bounded by $(1/e)G_K$.

We note that in deriving the above maximal throughput bound in (2.14), we have only utilized the uniform distribution associated with allocating retransmissions and the conditional distribution of new arrivals. Otherwise, the distribution of $A_n$ can be chosen arbitrarily and need not be a Poisson distribution.

To note the evolution of the mean number of period collisions, we incorporate (2.13) into relation (2.8). We then obtain for $n > 1$, noting that $N_n^{(j)} = T_n^{(j)} + A_n^{(j)} - S_n^{(j)}$, we have

$$E \{ S_{n+1} | N_n^{(j)} \} = E(A_n^{(j)}) - E \left( \frac{1}{K} \right)^{N_n^{(j-1)}},$$

(2.15a)

and therefore

$$E(N_n^{(j+2)} - N_n^{(j+1)} - E(A_n^{(j+2)}) - E \left( \frac{1}{K} \right)^{N_n^{(j-1)}}.$$  

(2.15b)

Also, since $R_n^{(j)} = R_n^{(j)} + A_n^{(j+1)} - S_n^{(j+1)}$, we have for $n > 1$

$$E(R_n^{(j+1)}) = E \left[ (R_n^{(j)} + A_n^{(j+1)}) \left( 1 - \left( \frac{1}{K} \right)^{R_n^{(j+1)} - 1} \right) \right].$$

(2.16)

Note in (2.15) that as the total number of period transmissions $N_n^{(j)}$ increases the throughput rapidly decreases and subsequently $E(N_n^{(j+2)} - E(N_n^{(j+1)}) \approx E(A_n^{(j+2)})$.

It can be verified that under an uncontrolled GRA access-control procedure (as for the SA procedure), the underlying channel state sequences are transient, yielding an unbounded limiting average packet delay ($D = \infty$). To stabilize the GRA channel, we are assuming the incorporation of a control function that rejects new packet arrivals within certain periods. This control function is characterized by the binary control variables $(U_n, n > 1)$, defined as follows. The 0–1 control variable $U_n$ will be determined causally by the past values of $(R_n, U_n)$. The variable $U_n, n > 1$, is set equal to 1 if all
new arrivals during the nth period are to be rejected. It is set equal to 0 if no packets are rejected during the nth period. The variables \( \{ U_n, n \geq 1 \} \) are directly incorporated in (2.9), (2.10) to determine the transition probability functions of the GRA state process, as shown in Fig. 1. In particular, the control variables induce the controlled arrival variables \( A_n(i) = (1 - U_n) A_n(i), n \geq 1, j = 1, 2, \ldots, K, \)

\[
\tilde{A}_n = \sum_{j=1}^{K} \tilde{A}_n(j).
\]

When the channel is controlled, (2.11)–(2.16) still apply if we replace the arrival variable \( A_n \) by the controlled arrival variable \( \tilde{A}_n \). In particular, we obtain from (2.16) the conditional means of \( R_{n+1} \),

\[
\tilde{R}_m(i) = i \left[ 1 - \left( 1 - \frac{1}{K} \right)^{i-1} \right], \quad (2.18a)
\]

and

\[
\tilde{R}_0(i) = \left[ 1 - e^{-\lambda} \left( 1 - \frac{1}{K} \right)^{i-1} \right] + K \lambda \left[ 1 - e^{-\lambda} \left( 1 - \frac{1}{K} \right) \right].
\]

\[
(2.18b)
\]

Proof: (2.18a) follows from (2.16) when we set \( A_{n+1} = 0 \). Equation (2.18b) is obtained by evaluating the expectation in (2.16) with \( A_{n+1} \) governed by a Poisson distribution with mean \( K \lambda \) and setting \( R_n = i \).

Functions \( \tilde{R}_d(i) \) and \( \tilde{R}_d(i) \) given by (2.18) are shown in Fig. 2. Note that \( \tilde{R}_d(i) - \tilde{R}_d(i), i > 0, \) for \( \lambda = 0 \). For any \( \lambda > 0 \), we observe \( \tilde{R}_d(i) \) to be a monotonically increasing function of \( i \) with a slope monotonically increasing to a rather constant value within \( 0 < i < 2K \). For large \( i \), \( R_0(i) \approx i \), as \( \lim_{i \to \infty} i^{-1} \tilde{R}_0(i) = 1 \). We further observe that \( \tilde{R}_d(i) - \tilde{R}_d(i) \approx K \lambda \) for \( i > K \) and that

\[
K \lambda (1 - e^{-\lambda}) = \tilde{R}_0(0) < \tilde{R}_0(i) < i + K \lambda. \quad (2.19)
\]

By setting \( \lambda = 0 \) in (2.19), we also note that

\[
0 < \tilde{R}_1(i) < i. \quad (2.20)
\]

The lower bound in (2.19) clearly specifies the average number of collisions among newly arriving packets during any period.

**Packet Delay Analysis**

The computation of the average packet waiting time (or similarly the distribution or any other moment of its waiting time) is now presented following the procedure used in [1] and briefly summarized as follows. We assume that an appropriate control function \( U_n = U(\cdot) \) is chosen so that the Markov state sequences \( Z \) and \( R \) are irreducible positive-recurrent. The transition probability functions and stationary distributions of \( Z \) and \( R \) are denoted by \( \{ P_{Z,i}(i,j) \}, \{ P_{R,i}(i,j) \}, \) respectively. We set \( N(Z_n, Z_{n+1}) \) as the number of newly admitted packets in \( B_n \) conditioned on \( (Z_n, Z_{n+1}) \). We let \( W(Z_n, Z_{n+1}) \) denote the sum of the waiting-time components of all packets transmitted during \( B_n \), given \( (Z_n, Z_{n+1}) \). The waiting-time component of such a packet is zero if the packet is successfully transmitted in \( B_n \) and otherwise equal to the number of slots between the packet's instants of transmission in \( B_n \) and retransmission in \( B_{n+1} \). The functions \( N(\cdot) \) and \( W(\cdot) \) are time-homogeneous functions of \( (Z_n, Z_{n+1}), n \geq 1 \). Considering nondegenerate controls that yield a packet rejection probability \( (P_R) \) less than one, we have

\[
\lim_{M \to \infty} \sum_{n=1}^{M} N(Z_n, Z_{n+1}) = \infty \quad (2.21)
\]
with probability one. Subsequently we can write
\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} W_n}{N} = \lim_{M \to \infty} \frac{\sum_{n=1}^{M} W(Z_n, Z_{n+1})}{\sum_{n=1}^{M} N(Z_n, Z_{n+1})}
\]
with probability one. We now apply a Markov ratio limit theorem (see [9, p. 91, Th. 1] and [11) to the vector Markov chain \( \{Y_n, n \geq 1\} \) where \( Y_n = (Z_n, Z_{n+1}) \). The latter is an irreducible positive-recurrent Markov chain with the stationary distribution \( \{\pi(i,j)\} \), where \( \pi(i,j) = \pi_z(i)P_z(i,j) \). We conclude that
\[
\bar{W} = \frac{E[W(Z_n, Z_{n+1})]}{E[N(Z_n, Z_{n+1})]}.
\] (2.22)
where
\[
E[W(Z_n, Z_{n+1})] = \sum_{i,j} W(i,j)\pi(i,j)
\]
\[
E[N(Z_n, Z_{n+1})] = \sum_{i,j} N(i,j)\pi(i,j).
\]
For the controlled GRA channel we have
\[
N(Z_n, Z_{n+1}) = \tilde{A}_n
\] (2.23)
and
\[
W(Z_n, Z_{n+1}) = [KR_n^{(1)}(K - 1)R_n^{(2)} \cdots \cdots R_n^{(k)}] + PR_n
\]
\[+ [T_n^{(2)} + 2T_n^{(3)} + \cdots + (K - 1)T_n^{(k)}].
\] (2.24)
The first term in (2.24) presents the overall delay of packets colliding in \( B_n \) from their instants of collision to the instant of termination of \( B_n \). The second term, \( PR_n \), expresses the overall delay of the \( R_n \) packets colliding in \( B_n \) over the \( P \) slots between \( B_n \) and \( B_{n+1} \). The third term yields the overall delay of these \( R_n \) packets between the starting instant of \( B_{n+1} \) and the following instants of retransmissions of these packets.

In equilibrium (applying probability measure \( \pi(i,j) \)), assuming the controlled channel to serve (eventually successfully transmit) all admitted packets, we have \( \lim_{n \to \infty} E(A_n) = \lim_{n \to \infty} E(S_n) \) and therefore
\[
E(N(Z_n, Z_{n+1})) = \lim_{n \to \infty} E(S_n) = Ks.
\] (2.25)
Also by symmetry, since arriving and retransmitting packets are uniformly distributed over \( B_n \), we obtain
\[
E(R_n^{(1)}) - E(T_n^{(1)}) - K^{-1}E(R_n).
\] (2.26)
Using (2.23)–(2.26), we obtain by (2.22) the formula for computing the limiting average packet waiting time \( \bar{W} \) in a GRA channel in terms of the steady-state means of \( \{R_n, n \geq 1\} \) and \( \{S_n, n \geq 1\} \), \( \bar{R} \) and \( Ks \), respectively.

**Theorem 2**: For a controlled GRA channel represented by an irreducible positive-recurrent state sequence \( Z \), the limiting average packet waiting time \( \bar{W} \) is given by
\[
\bar{W} = [P + K] \bar{R}.
\] (2.27)

We note that \( \bar{R} \) represents the average number of retransmissions/admitted packet. Eq. (2.27) thus indicates that a retransmitted packet experiences an average delay of \( P + K \) slots for each of its retransmissions as expected. A procedure for the optimal control of a GRA channel incorporating both packet average time delay and probability of rejection as indices of performance is developed and studied in the next section.

**III. Optimal Dynamic Control of a GRA Channel**

**The Optimal Control Problem**

The dynamics of the GRA channel, governed by control sequence \( \{U_n, n \geq 1\} \), are described in Fig. 1 and (2.8) (2.10) in terms of the underlying state sequence \( Z \). Assuming the Markov chain \( \{(R_n, U_n), n \geq 1\} \) to be irreducible positive-recurrent as is the case for all our applications of interest, the stationary probabilities of the latter chain have been noted to determine the major indices of performance. The average packet waiting time \( \bar{W} \) is the first measure of performance of interest. It is given by (2.27), thus depending on the latter Markov chain only through \( \bar{R} \) and throughput \( s \). The second measure of performance of interest here is the probability of packet rejection \( PR \), directly related to the channel throughput \( s \) by (2.4). We wish to obtain the control sequence \( U = \{U_n, n \geq 1\} \) that will yield the minimum value of packet average waiting time (or delay), while providing an appropriate prescribed value of packet rejection probability \( P_R \) (or channel throughput \( s \)).

Note that rejected packets can be assumed to be lost or to try again to access the channel following an appropriate random delay. The latter is then assumed to follow an exponential distribution so that the point process of new arrivals is still a Poisson process. (This process will still have intensity \( \lambda \) if we consider a network with a large number of small terminals, each blocked until its stored packet is successfully transmitted.) Since the probability \( P_R \) for most applications will be very small (see latter curves), the precise rejection-reenter mechanism is not important for the present analysis.

Assuming causal observations of the controlled channel state sequence \( Z \) are available to the controller \( U \), only Markov sequence \( \{(R_n, U_n), n \geq 1\} \) needs to be causally observed since measures \( \bar{W} \) and \( P_R \) are considered. (In many actual situations only the variable indicating whether collisions occur in a slot and not the number of collisions occurring there can be observed. We will note later in this section that a scheme using the former observations exhibits a performance very similar to that obtained when the latter observations are assumed.) The set of admissible control functions \( \mathcal{U} \) is not constrained
and includes all deterministic and randomized binary functions operating on all past observations of \{(R_n, U_n)\}. Thus the control variable operating at the nth period \(U_n\) is expressed in terms of a function \(f_a(\cdot)\), where \(U_n = f_a(R_{1,n-1}, U_{1,n-1})\). \(R_{1,n} = \{ R_m, 1 \leq m \leq n \}\), \(U_{1,n} = \{ U_m, 1 \leq m \leq n \}\), and \(U_n \in \{0,1\}\). (Note, however, that only binary control functions are considered and that all rejected packets are required to retransmit in the next period by the GRA protocol.) Since only finite average packet delays are of interest, we need to consider only the subset of control disciplines \(\mathcal{D}_E \subset \mathcal{A}\) that result in a positive-recurrent controlled Markov chain \(\{R_n, n \geq 1\}\) with finite mean \(\bar{R} < \infty\) where 

\[
\bar{R} = \lim_{n \to \infty} E(R_n) = \lim_{n \to \infty} N^{-1} \left\{ \sum_{n=1}^{N} R_n \right\} \tag{3.1}
\]

and a packet probability of rejection \(P_R\) given by 

\[
P_R = \lim_{N \to \infty} N^{-1} E \left\{ \sum_{n=1}^{N} I(U_n=1) \right\} \tag{3.2}
\]

The countable number of feasible control functions \(u \in \mathcal{D}_E\) induce the set \(\mathcal{D}_R\) of values of attainable rejection probabilities. Thus 

\[
\mathcal{D}_R = \{ p : p = P_R(u), u \in \mathcal{D}_E \} \tag{3.3}
\]

where \(P_R(u)\) denotes the rejection probability resulting under control function \(u\). The minimal attainable rejection probability is given by 

\[
P_R^0 = \inf \{ p : p \in \mathcal{D}_R \} \tag{3.4}
\]

Given any maximal allowable value of rejection probability \(P_R, P_R > P_R^0\), we wish to obtain the minimal attainable average packet waiting time \(\bar{W}(P_R)\), given by 

\[
\bar{W}(P_R) = \inf \{ \bar{W}(u) : P_R(u) < P_R \} \tag{3.5}
\]

where \(\bar{W}(u)\) denotes the average packet waiting time obtained when control function \(u\) is used. An optimal control function attaining waiting time \(\bar{W}(P_R)\) and yielding rejection probability equal to at most \(P_R\) is denoted by \(u^*_R(P_R)\).

To obtain points on the optimal delay-throughput curve of (3.5), let \(P_R\) vary between \(P_R^0\) and 1, we can in turn derive the function \(\phi_P(\beta)\), as \(\beta\) varies in \([0, \infty)\) where 

\[
\phi_P(\beta) = \inf_{u \in \mathcal{D}_E} \{ \bar{W}(u) + \beta P_R(u) \} \tag{3.6}
\]

Alternatively since \(\bar{W}\) is given by (2.27), we can derive a function \(\phi_R(\beta)\) by 

\[
\phi_R(\beta) = \inf_{u \in \mathcal{D}_E} \{ \bar{R}(u) + \beta P_R(u) \} \tag{3.7}
\]

where \(\beta > 0\) and \(\bar{R}(u)\) denotes the limiting average number of retransmissions (3.1) under control function \(u\). Given \(\beta\), let \(u^*_R(\beta)\) denote a control function attaining the minimum at \(3.7\). We have thus obtained the following result.

Proposition 2: For a GRA channel, points on curve \(\phi_R(\beta)\) of (3.7), \(\beta > 0\), correspond to points on the delay-throughput curve \(\bar{W}(P_R)\) of (3.5), \(P_R > P_R^0\). Thus for each \(\beta > 0\), 

\[
\bar{W}(u^*_R(\beta)) = \bar{W}(P_R^*(\beta)) \tag{3.8}
\]

where 

\[
P_R^*(\beta) = P_R(u^*_R(\beta)) \tag{3.9}
\]

so that 

\[
u^*_R(P_R(\beta)) = u^*_R(\beta) \tag{3.10}
\]

and \(1 > P_R^*(\beta) \geq P_R^0\).

Proof: For each fixed \(\beta > 0\), the solution \(u^*_R(\beta)\) to (3.6) yields \(\bar{W}(P_R)\) of (3.5) for \(P_R = P_R(u^*_R(\beta))\), since by (3.6) control function \(u^*_R(\beta)\) attains the minimal average waiting-time value \(\bar{W}(u)\), considering all control functions \(u\) yielding \(P_R(u) = P_R(u^*_R(\beta))\) for example, see [17, p. 220, Th. 1]. By (2.27) for \(\bar{W}\), we note that \(\bar{W}\) depends on \(R\) and \(P_R\) (through \(s\), see (2.4)). Therefore, for any given \(P_R, \bar{W}\) is minimized if and only if the corresponding \(R\) is minimized. Hence curve \(\phi_R(\beta)\) yields curve \(\phi_P(\beta)\) and the corresponding points on curve \(\bar{W}(P_R)\).

Proposition 2 indicates that for each fixed \(\beta\), a solution to minimization problem (3.7) will yield the minimal delay value \(\bar{W}(P_R)\) under the corresponding probability of rejection \(P_R\) given by (3.9). As \(\beta\) increases, a solution to (3.7) will represent a scheme with a nonincreasing rejection probability \(P_R^*(\beta)\) and a nondecreasing average waiting-time function \(\bar{W}(\beta) = \bar{W}(u^*_R(\beta))\). It is readily noted that for a high enough \(\beta\) value, the solution to (3.7) yields the minimal rejection probability \(P_R^0\). For \(\beta = 0\), this solution induces \(P_R = 1\). Thus as \(\beta\) varies, the solutions to (3.7) yield points on the curve \(\bar{W}(P_R)\) and the corresponding optimal schemes at \(P_R\) values (3.9). For all practical purposes, the latter values generally represent a dense enough set of rejection probabilities in \([P_R^0, 1]\). Note also the role of \(\beta\) in (3.6) as a penalty cost for packet rejections. Thus (3.6) could have been specified as the primary objective function.

The optimal control problem (3.5) has thus been represented in the form of a Markov decision problem (3.7), described as follows. The stochastic process \(\{R_n, n \geq 1\}\) with state-space \(S = \{0,1,2,\cdots\}\) is controlled by the binary control sequence \(\{U_n, n \geq 1\}\), \(U_n \in \{0,1\}\). At time \(n\) corresponding to the end of the nth period \(B_n\), state \(R_n\) is observed and an action (or control) \(U_{n+1} = f(R_{1,n}, U_{1,n})\) is taken. Subsequently a cost \(C(R_n, U_{n+1})\) is incurred, and the next state of the process is chosen according to transition probabilities \(\{P_{U}(U_{n+1}|U_n)\}\). Thus the controlled process \(\{R_n, n \geq 1\}\) transition probability function satisfies 

\[
P \{ R_{n+1} = j | R_{1,n} = i, U_{1,n} = i, R_n = i, U_{n+1} = a \} = P_{ij}(a) \tag{3.11}
\]

where \(a \in A = \{0,1\}\). Under an expected average cost criterion (for example, [10]–[12]), an admissible control function \(u \in \mathcal{D}_E\) is chosen to minimize the long-run ex-
expected average cost/unit time. For control policy \( u \), the associated cost function is then given by

\[
\phi_u(i) = \limsup_{N \to \infty} N^{-1} \mathbb{E}_u \left( \sum_{n=1}^{N} C(R_n, U_{n+1}) | R_1 = i \right),
\]

(3.12)

for \( i \in \mathcal{S} \), where \( E_u \) indicates that the conditional expectation given \( u \) is used. A control function \( u^* \) is said to be average cost optimal if

\[
\phi_u(i) = \inf_{u \in \mathcal{U}_E} \phi_u(i), \quad \text{for all } i.
\]

(3.13)

Incorporating (3.1), (3.2) in (3.7), we obtain

\[
\phi_R(\beta) = \inf_{u \in \mathcal{U}_E} \lim_{N \to \infty} N^{-1} \mathbb{E}_u \left( \sum_{n=1}^{N} [R_n + \beta U_{n+1}] \right)
\]

(3.14)

since \( I(U_n+1) = U_n \), \( n \geq 1 \). Comparing (3.14) with (3.12), (3.13), we deduce the following result.

**Proposition 3:** The optimal control policy \( u^*_\beta \) yielding \( \phi_R(\beta) \) for a GRA channel, for any \( \beta > 0 \), is an average cost optimal control policy for the Markov decision process \( \{R_n, U_n\}, n \geq 1 \), under the long-run expected average cost/unit time measure (3.12) with an associated cost function \( C(R_n, U_{n+1}) \) given by

\[
C(R_n, U_{n+1}) = R_n + \beta U_{n+1}.
\]

(3.15)

Results from Markov decision theory are incorporated in the following analysis to obtain the structure of optimal policy \( u^*_\alpha \), under cost measure (3.14) or (3.12), (3.15). We note that the corresponding cost values become unbounded for control \( u \in \mathcal{U}_E \). At the same time, the simple control function \( u \in \mathcal{U}_E \) that rejects all arrivals clearly yields \( P_R(u_i) = 1, R(u_i) = 0 \), and subsequently

\[
\phi_u(i) - \phi_R(\beta) < \phi_u(i) - \beta.
\]

(3.19)

Therefore the search for an optimal control policy can be reduced to subclass \( \mathcal{U}_E \). The same conclusion will be observed to hold under cost measure \( V_u(i) \) when \( \alpha \) is taken to be close enough to 1.

We will first establish that an optimal policy \( u^*_\alpha \) in fact exists and that it is a stationary policy. The structure of the optimal policy will then be characterized. For that purpose we first consider the \( \alpha \)-discounted cost problem (3.17) and subsequently study the characteristics of the \( \alpha \)-optimal policies. The policy obtained as a limit of the \( \alpha \)-optimal policies is then shown to yield the desired optimal control function \( u^*_\alpha \).

It is well known that a stationary \( \alpha \)-optimal policy \( u^*_\alpha \) for (3.17) exists whenever cost function \( C(\cdot) \) is bounded (see [10], [16]). However, for our problem \( R_n \) and therefore \( C(\cdot) \) are unbounded. It has, however, been shown in [12] that an optimal stationary (deterministic) policy for the \( \alpha \)-discounted cost problem exists if the following conditions are met. The existence is required of an integer \( m \geq 1 \), a real-valued function \( g(\cdot) \) on \( \mathbb{S} \) with \( g(x) > 1 \) for all \( x \in \mathbb{S} \), and a real number \( b > 0 \) such that

\[
L = \sup_{x \in \mathbb{S}} \left( \max_{a \in A} C(x,a)g(x)^{-m} \right) < \infty
\]

(3.20)

and for all \( x, n = 1, 2, \cdots, m \),

\[
\max_{a \in A} \sum_{y \in \mathbb{S}} g(y)^\alpha P_y(a) < \left[ g(x) + b \right]^\alpha.
\]

(3.21)

We now verify that conditions (3.20), (3.21) hold for our problem and subsequently deduce the following result.

**Lemma 1:** For a controlled GRA channel, an optimal control policy \( u^*_\alpha \) exists and is stationary for the \( \alpha \)-discounted cost problem (3.17) where \( \alpha \in (0, 1) \). The minimal discounted cost \( V_u(i) \) is the unique solution to

\[
V_u(i) = \min_{a \in A} \left\{ C(i,a) + \alpha \sum_{j \geq 0} V_u(j) P_{y}(a) \right\}
\]

(3.22)

the functional equation of dynamic programming. Furthermore, \( u^*_\alpha \) is the policy that selects an action minimizing the right side of (3.22) for each \( i > 0 \).

**Proof:** The results follow by [12], once conditions (3.20), (3.21) are verified. For that purpose, we choose
The slope function $\Delta_i(a) = \overline{R}_{i+1}(a) - \overline{R}_i(a)$ first increases monotonically as $i$ increases to a value greater than 1. Then for large $i$ the slope function decreases monotonically to its asymptotic value of 1; $\lim_{i \to \infty} i^{-1} \overline{R}_i(a) = 1$. Therefore the slope $c$ in approximation (3.26) actually varies with $i$. Using (3.26) to compute $V_a(i,n)$ from the recursion (3.24), we note that $V_a(i,n)$ depends linearly on $i$. Then (3.26) can be reapplied in (3.24) to obtain $V_a(i,n+1)$ from $\{V_a(j,n), j > 0\}$. We find that $V_a(i,n)$ is given for $ac < 1$ by

$$V_a(i,n) = A(i,a,n) + \min_{a \in A} a[\beta - D(a,c,n)] \quad (3.27a)$$

where

$$D(a,c,n) = ad_2 \frac{1-(ac)^{n-1}}{1-ac}, \quad n > 2. \quad (3.27b)$$

Consider now the set of states $\{i : i \geq 2K\}$. We have noted above that $\Delta_i(a) > 1$ for $i > K$ and $a = 0, 1$. Furthermore,

$$P_j(a) = 0, \quad j < i - K, i > K, a = 0, 1, \quad (3.28)$$

since at most $K - 1$ successful transmissions can occur when more than $K$ transmissions are attempted. Therefore in carrying iteration (3.24) under assumption (3.26), we can incorporate the observation that

$$\sum_{j > 0} \overline{R}_j(a)P_j(a) = \sum_{j \geq K+1} \overline{R}_j(a)P_j(a) \quad (3.29)$$

when $i > 2K$. Consequently, in evaluating $V_a(i,n)$ by (3.24) for $i > 2K$, only functions $\{\overline{R}_j(a), j > K\}$ are involved. Since the latter functions have a slope larger than one, we conclude that $D(a,c) = \lim_{n \to \infty} D(a,c,n)$ becomes arbitrarily large for $\alpha$ close enough to 1; subsequently $[\beta - D(a,c)]$ becomes negative, resulting in an optimal control function satisfying the following property.

**Lemma 2:** For a GRA channel and for any $\beta > 0$, there exists an integer $M < \infty$ such that the $\alpha$-discounted optimal stationary control $u_{\beta}^*(a,i)$ satisfies

$$u_{\beta}^*(a,i) = 1 \quad (3.30a)$$

for $i > M$ and for all $\alpha$ such that $1 > \alpha > \alpha_p$ for some $\alpha_p$. The stationary optimal control policy $u_{\beta}(i)$ similarly satisfies

$$u_{\beta}(i) = 1, \quad i > M. \quad (3.30b)$$

Furthermore, $M < 2K$.

For $0 < i < 2K$, the appropriate corresponding slope $c$ increases monotonically from a small value, so that $D(a,c,n)$ is replaced by a function $D(a,i,n)$ that is increasing monotonically in $i$. Thus letting $n \to \infty$, the recursion (3.24) yields a function $V_a(i)$ given by

$$V_a(i) = A(i,a) + \min_{a \in A} a[\beta - D(a,i)] \quad (3.31)$$

The function $D(a,i)$ is increasing monotonically in $i$ for $i < M$, where $M$ is an appropriate finite integer. For $i > M$ and any $\beta > 0$, we have

$$D(a,i) > \beta, \quad a > \alpha_p, i > M. \quad (3.32)$$
for some \( \alpha_0 \) close enough to 1 so that

\[
\lim_{a \to 1} D(\alpha, i) < \infty, \quad \text{if and only if } i < M. \tag{3.33}
\]

Properties (3.32), (3.33) are incorporated in Lemma 2. Using expressions (3.31)-(3.33), we can thus deduce the character of the optimal control policy as summarized in Theorem 4.

**Theorem 4:** For \( \beta > 0 \), the optimal control function \( u_\beta^* \) attaining \( \phi_\beta(\beta) \) is characterized as a single-threshold control function \( u_\beta^*(i) \) given by

\[
u_\beta^*(i) = u_\beta(i - K_\beta(\beta)) = \begin{cases} 1, & \text{if } i > K_\beta(\beta) \\ 0, & \text{if } i < K_\beta(\beta) \end{cases} \tag{3.34}
\]

where \( u_\beta(\cdot) \) is the unit-step function. The threshold \( K_\beta(\beta) \) satisfies

\[
0 < K_\beta(\beta) < M, \quad \text{for } \beta_0 < \beta < \beta_{\text{max}} \\
K_\beta(\beta) = M, \quad \text{for } \beta = \beta_{\text{max}} \\
K_\beta(\beta) = 0, \quad \text{for } \beta < \beta_0\tag{3.35}
\]

where

\[
\beta_{\text{max}} = \lim_{a \to 1} D(\alpha, M - 1) \tag{3.36a}
\]

\[
\beta_0 = K_\lambda[1 - \exp(-\lambda)] \tag{3.36b}
\]

and \( M < 2K \) is the integer appearing in Lemma 2.

**Proof:** (3.34) follows by (3.31) from the monotonicity of \( D(\alpha, i) \) for \( i < M \), and from (3.32) or (3.33) for \( i > M \). These relationships also yield (3.35) for \( \beta > \beta_0 \). To prove \( K_\beta(\beta) = 0 \) for \( \beta < \beta_0 \), so that \( u_\beta(i) = 1 \) for each \( i \), we note that if \( c_n(a) \) denotes the sum of the average costs over the \( n \)th and \((n+1)\)st periods when \( U_n = a \), then

\[
c_n(0) - c_n(1) > \beta_0 - \beta > 0
\]

since \( \beta_0 \) expresses the average number of collisions during a period due to new arrivals. Therefore we attain \( \phi_\beta(\beta) = \beta \) by rejecting all transmissions, and \( K_\beta(\beta) = 0 \). \( \square \)

Theorem 4 characterizes the stationary optimal control policy \( u_\beta(i) \) as a simple single-threshold scheme. (It is readily shown to yield a positive-recurrent Markov chain.) For \( \beta < \beta_0 \), the optimal scheme rejects all arrivals yielding a packet rejection probability \( P_R \) equal to one, and thus an average waiting time \( \bar{W} = 0 \). As \( \beta \) is increased, the threshold \( K_\beta(\beta) \) of the associated optimal single-threshold scheme increases, yielding lower values of \( P_R \), the minimal \( R \), and the corresponding minimal average packet waiting time \( \bar{W} \). However, there is a saturation effect. If \( \beta \) exceeds \( \beta_{\text{max}} \), the same optimal single-threshold scheme is obtained with threshold \( K_\beta(\beta) = M \). The latter scheme yields a packet rejection probability \( P_R = P_R^M \) that is clearly equal to the minimal attainable \( P_R \) value. Using Proposition 2, the optimal control scheme has thus been shown to be governed by the following characteristics.

**Corollary 2:** For a GRA channel, there exists a stationary optimal control scheme \( \hat{u}_\beta(i) \) that solves (3.7) for any \( \beta > 0 \) and yields the minimal \( \bar{R} \) and \( \bar{W} \) values under a corresponding prescribed maximal packet rejection probability \( p \) where \( P_R^0 < p = P_R(u_\beta^*) < 1 \). Such a scheme assumes a single-threshold structure given by

\[
\hat{u}_\beta(i) = \begin{cases} 1, & \text{if } i > K(\beta) \\ 0, & \text{if } i < K(\beta) \end{cases} \tag{3.37}
\]

The threshold \( K(\beta) \) increases monotonically from \( K(1) = 0 \) to \( K(0) = M \) as \( p \) is decreased from 1 to \( P_R^0 \). The rejection probability \( P_R^0 \) attained by the single-threshold scheme with threshold \( M \) is the minimum attainable such probability for optimization problem (3.5).

It is interesting to observe that as the threshold \( K \) of the single-threshold scheme (3.37) is increased from 0 to \( M \), the packet rejection probability and average waiting time are respectively decreased and increased. However, a further increase of the threshold above \( M \) in such a scheme only increases \( P_R \) while also increasing \( \bar{W} \) and should therefore be avoided. This phenomenon is explained by noting that for such a scheme with a threshold higher than \( M \), the time gained to threshold upcrossing is more than offset by the extra time required for threshold downcrossing.

**Performance Results**

The performance curve \( \bar{W}(p) \) of the optimal single-threshold scheme can now be computed using (2.27), (2.28), with the control function (3.37) incorporated in Fig. 1 to yield the necessary statistics (\( s \) and \( R \)). The latter have been computed by a straightforward simulation of the Markov chain \( \{R_n, n \geq 1\} \) under the control function (3.37). The resulting performance curves are shown in Figs. 3 and 4 for \( K = 12 \) and in Figs. 5 and 6 for \( K = 9 \). In both cases, we set \( R = P = 12 \). Average packet delay (\( D \)) versus packet probability of rejection (\( P_R \)) curves are shown in Figs. 3 and 5. For a fixed value of \( \lambda, \lambda = 0.2, 0.3, 0.4 \), we note the variation of \( D \) versus \( P_R \) as the threshold \( K_1 \) of a single-threshold scheme is increased from \( K_1 = 3 \) to \( K_1 = 30 \). The characteristics of the optimal schemes as stated in Theorem 3 are well demonstrated in these figures. The minimal probability of rejection for the GRA channel with \( K = 12 \) is effectively zero for \( \lambda = 0.1 \). Also \( P_R^0 = 0.004 \) for \( \lambda = 0.3 \) and \( P_R^0 = 0.114 \) for \( \lambda = 0.4 \). For \( K = 9 \), \( P_R^0 = 0.002 \) for \( \lambda = 0.3, P_R^0 = 0.1 \) for \( \lambda = 0.4 \), and \( P_R^0 \) is very small for \( \lambda < 0.2 \). (Note that as the Markov chain simulation is run for a large but finite number of slots, no threshold crossings would occur for low \( \lambda \) values and high \( K_1 \) values, thus accounting for the form of the curves shown for low \( \lambda \) values.) For \( K = 12 \), the minimal \( P_R \) values (\( P_R^0 \)) are attained at thresholds \( K_1 \approx 16 \) and \( K_1 = 12 \) for \( \lambda = 0.3 \) and \( \lambda = 0.4 \), respectively. For \( K = 9 \), the thresholds attaining \( P_R^0 \) are \( K_1 \approx 18 \) and \( K_1 \approx 9 \) for \( \lambda = 0.3 \) and \( \lambda = 0.4 \), respectively. It can be noted that such a scheme includes a threshold \( K_1 = K \) attaining an excellent performance (\( D \) versus \( P_R \)) over the whole range of network traffic intensities (including generally any \( \lambda \) with \( 0 < \lambda < 0.4 \) yielding a rejection probability not higher than \( P_R^0 > 0.1 \). We further note that a threshold value \( K_1 = M \) yielding a minimal probability of rejection would not cause an
Fig. 3. Delay versus probability of rejection curves for GRA channel with $K = R = P = 12$ under single-threshold control scheme with threshold $K_1$. Constant $-\lambda$ curves with parameter $K_1$ and performance points $\triangle$ for scheme $C$ with threshold $K_2$ and points $\square$ for scheme $C_2$ for $\lambda = 0.3$.

Fig. 4. Delay versus throughput curves for GRA channel with $K = R = P = 12$ under single-threshold schemes with threshold $K_1$ and parameter $\lambda$. Constant $-\lambda$ curves for $\lambda = 0.3$ (---) and $\lambda = 0.4$ (-----), and performance points $\triangle$ for scheme $C$ with threshold $K_2 = 3$.

Fig. 5. Delay versus probability of rejection curves for GRA channel with $K = 9$, $R = P = 12$, under single-threshold scheme with threshold $K_1$. Constant $\lambda$ curves with parameter $K_1$.

Fig. 6. Delay versus throughput curves for GRA channel with $K = 9$, $R = P = 12$, under single-threshold schemes with threshold $K_1$ and parameter $\lambda$ and constant $-\lambda$ curves for $\lambda = 0.3$ (-----) and $\lambda = 0.4$ (-----).
average packet delay much higher than a threshold value yielding a much higher \( P_R \) value. Therefore it is generally preferable to assign a threshold value of \( K_1 = M \) to the GRA channel controller. For example, for \( K = R = 12 \), and threshold values \( K_1 = 3, 6, 12 \), we obtain \( P_R \) values of \( P_R = 0.162, 0.062, 0.01 \) and delay values of \( D = 22, 25, 29 \) slots, respectively.

The associated delay \( (D) \) versus throughput curves are shown in Figs. 4 and 6. Note that for \( K = 12 \), the maximal throughput value of \( s = e^{-1} \) is attained at \( \lambda = 0.8 \) by a scheme with threshold \( K_1 = 6 \), yielding therefore (at this traffic value) a rejection probability value of \( P_R = 1 - \lambda e^{-1} \approx 0.54 \). For \( K = 9, s = e^{-1} \) is attained at \( \lambda = 0.6 \) by a scheme with \( K_1 = 7 \), yielding \( P_R = 1 - \lambda e^{-1} \approx 0.39 \). It is noted that over the practical throughput range \( 0 < s < 0.3 \), the average packet delay varies slowly and nearly linearly from \( D = 13(= R + 1) \) at \( s = 0 \), via \( D = 20 \) at \( s = 0.2 \), to only \( D = 25 \) at \( s = 0.3 \), at both \( K = 12 \) and \( K = 9 \). GRA schemes and any threshold value \( K_1 \) with \( 3 \leq K_1 \leq K \).

**Other Control Schemes for a GRA Channel**

When the GRA discipline is governed by a distributed control procedure applied over a broadcast channel, the process \( \{ R_n, n > 1 \} \) often cannot be observed by the individual terminals. The terminals generally can only observe in each slot whether a successful transmission or a collision has occurred. In the latter case, the terminal obtains no information about the number of collisions involved. Thus the process observed by each terminal is given by \( \{(S_n, C_n), n > 1\} \), where \( S_n \) denotes the number of successful transmissions within the \( n \)th group \( B_n \) and \( C_n \) gives the total number of slots in \( B_n \) experiencing collisions, given by

\[
C_n = \sum_{j=1}^{K} C_n^{(j)} = \sum_{j=1}^{K} I(N_n^{(j)} > 2) \tag{3.38}
\]

for \( n > 1 \) where \( \{N_n^{(j)}\} \) is given by (2.9c). Note that \( \{(S_n, C_n), n > 1\} \) is not a Markov chain. An optimal control procedure incorporating the latter observation chain can be developed in a manner similar to that presented above. However, because the special character of the underlying state sequence, we can readily make the following observations.

Note that \( R_n \gg 2C_n \). Within the range of acceptable packet delay values we further expect each collision to involve an average number of transmissions that is very close to 2 and lower than 3. Thus we should have \( R_n \approx rC_n \) within this range, with \( r \approx 2 \). When \( R_n > rC_n \), the number of group collisions rapidly increases, and therefore higher and generally unacceptable packet delay values are obtained. Therefore, estimating \( \{R_n, n > 1\} \) by \( \{\hat{R}_n = rC_n, n > 1\} \), we can employ the optimal single-threshold scheme developed above. The latter scheme denoted by \( \hat{C} \) now uses observations of \( \{C_n, n > 1\} \) and a threshold \( K_2 \). We thus expect this scheme to exhibit a delay-throughput curve very close to that obtained by the optimal scheme that uses \( \{R_n\} \) observations and threshold \( K_1 \) and serves as a lower bound to the performance curve of \( \hat{C} \) with \( K_2 = r^{-1}K_1 \).

Performance points for a single-threshold scheme \( C \) are shown in Figs. 3, 4 for \( K = R = P = 12 \). The results completely verify the above observations. The optimal performance points for \( C \) all lie on the lower-bound performance curves for the scheme using \( \{R_n\} \) observations. Furthermore, the scheme \( C \) with threshold \( K_2 = K_1/2 \) attains performance curves very close to those obtained by the optimal scheme using \( \{R_n\} \) observations and a threshold \( K_1 < M \). However, when the latter scheme has utilized a threshold \( K_1 > M \), we note that scheme \( C \) would utilize a threshold \( K_2 \geq 2K_1 \). The latter situation represents a range of undesirable threshold values since schemes using lower threshold values yield lower packet delays under the same \( P_R \) values (see Fig. 3). We thus conclude that a single-threshold scheme using \( \{C_n\} \) observations operates as well as the optimal single-threshold scheme using \( \{R_n\} \) observations within the acceptable range of \( D - P_R \) values.

The methods presented and used in this paper can be applied to study a variety of other related GRA access-control disciplines. For example, in certain applications we might wish to reject certain colliding packets rather than new transmissions. We can thus study a GRA scheme with a 0–1 control function \( \{U_n\} \) that uses \( \{R_n\} \) or \( \{C_n\} \) observations and rejects at appropriate times all collisions within the corresponding period. The performance analysis for such a scheme follows that presented in Sections II and III. In particular, we can note that the associated Markov decision problem involves now the cost function \( C(R_n, U_n) = R_n[1 + \beta U_n] \). The resulting single-threshold scheme denoted by \( C_2 \) is rapidly shown to have similar performance characteristics to those indicated in Theorem 4 and Corollary 2. Performance points for a \( C_2 \) scheme for \( \lambda = 0.3, K = R = P = 12 \) are shown in Fig. 3. The average packet delay value here incorporates both successful and rejected transmissions. We note that this scheme yields lower packet delay values at higher rejection probabilities (\( P_R > 0.05 \)) when compared with the previous scheme. For rejection probabilities \( 0.014 < P_R < 0.05 \), comparable packet delays are attained by both schemes. Scheme \( C_2 \), however, yields a minimal probability of rejection \( P_R = 0.014 \), while the previous scheme yields \( P_R = 0.004 \). The latter is thus preferable at lower \( P_R \) values.

We also note that the delay-throughput performance curves presented here for a controlled GRA channel are similar to the corresponding curves obtained under a controlled SA procedure.

**IV. CONCLUSIONS**

We have presented and studied group random-access control disciplines for a multi-access communication channel. A GRA scheme uses only certain channel time
periods during which some network terminals attempt to transmit their information-bearing packets on a random-access basis. The channel can thus be utilized at other times to grant access to other terminals or message types.

To stabilize the GRA channel, an appropriate dynamic control procedure is applied. The state of the underlying channel state sequence is observed by each terminal, and accordingly within certain periods no new transmissions are allowed. During these periods, newly arriving packets are thus rejected. The performance of a dynamically controlled GRA channel is characterized in terms of the average packet delay (D) and the packet probability of rejection (PR) or the network throughput(s). The optimal control policy is derived yielding the minimal average packet delay under a prescribed maximal rejection probability. This policy is characterized by studying the associated Markov decision process. The resulting controller employs a single threshold $K_1$, with which the recent number of period collisions is compared. Subsequently new packets in the present period are either accepted or rejected. A Markov ratio limit theorem is used to evaluate the packet average waiting time function.

The performance results presented here demonstrate the low packet delay obtained under a GRA channel over the whole acceptable range of traffic intensities. We further note that a controller using a threshold value $K_2 = M$ yielding the minimal probability of rejection is many times a good choice. Furthermore, if only the sequence $\{C_n, n > 1\}$ indicating the number of colliding slots within a group can be observed as is generally the case for distributed control broadcast channels, we have shown that a corresponding single-threshold scheme $\tilde{C}$ with threshold $K_2 \approx K_1 / 2$ yields a nearly optimal delay-throughput performance. Other control schemes are noted to be governed by similar characteristics and analyzed using similar methods. As for a slotted-ALOHA (SA) random-access procedure, the GRA scheme is shown to allow a maximal throughput of $e^{-1}$. We also note the delay-throughput characteristics of a controlled GRA channel to be similar to those of an appropriately controlled SA channel. A GRA access-control procedure, however, allows for a much higher degree of dynamic and efficient utilization of a multi-access channel that utilizes integrated random-access, reservation, and fixed access-control procedures [1],[18], or utilizes the GRA scheme only to provide channel access to certain protocol packets.

**APPENDIX**

**Proof of Theorem 3**

The proof uses Lemma 1 and Corollary 1 and follows a similar procedure to that used in [14]. Considering the $\alpha$-discounted cost problem, we readily verify (noting that $u^{*}_{\alpha}$ yields a positive-recurrent controlled chain) that

$$\lim_{\alpha \to 1} (1 - \alpha)V_\alpha(i) = \lim_{\alpha \to 1} (1 - \alpha)V_\alpha(0).$$

(A1)

Furthermore, using (3.24), we will observe that there exists a stationary policy $\hat{u}$ and an increasing sequence $(\alpha_n)$ with $\alpha_n \uparrow 1$ such that

$$\phi_\alpha(i) = \lim_{\alpha \rightarrow \infty} (1 - \alpha)V_\alpha(i)$$

for all $i > 0$. By an Abelian theorem (see [14],[15]) and conditions (3.20), (3.21) for $i > 0$ and $\alpha \in \mathbb{R}$, we obtain

$$\phi_\alpha(i) > \lim_{\alpha \rightarrow 1} (1 - \alpha)V_\alpha(i).$$

(A3)

Therefore, from relations (A1)–(A3) and the sequence $(\alpha_n)$ used in (A2), we conclude that for any $u \in \mathbb{R}$ and $i > 0$ we have

$$\phi_u(i) > \phi_\alpha(i) = \phi_\alpha(0)$$

(A4)

$$\phi_u(i) = \phi_{\alpha_n}(i) = \phi_{\alpha_n}(0)$$

(A5)

$$u^* = \hat{u}.$$  

(A6)

More explicitly, we will note (see Lemma 2) that by iterating (3.24) and observing the resulting policies that, for a large enough value of $\alpha$, $\alpha \geq \alpha_u$, the stationary optimal control function $u^{*}_{\alpha}(\alpha)=\hat{u}$ is such that $\alpha_u(i)=1$ for any $i$ greater than some finite integer $M$. Therefore we can choose a sequence $(\alpha_n)$ with $\alpha_n \uparrow 1$ yielding a stationary control policy $u^{*}_{\alpha_n}(\alpha_n)=\hat{u}$ for each $n$ where $\hat{u}$ is the scheme derived from $\alpha_u$ as $\alpha \rightarrow 1$. Therefore using (A3) we obtain for $u \in \mathbb{R}$ and $i > 0$

$$\phi_u(i) > \lim_{\alpha_n \rightarrow 1} (1 - \alpha_n)V_{\alpha_n}(i),$$

(A7)

where the equality in (A7) follows from an Abelian theorem ([15]) stating that

$$\lim_{\alpha_n \rightarrow 1} \sum_{n=1}^{N} (1 - \alpha_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^{N} C_n.$$  

Therefore $u^* = \hat{u}$.

**References**


Some Recent Results on Characterizations of Measures of Information Related to Coding

JÁNOS ACZÉL

Abstract—Essential properties of entropies and other measures of information are arrived at on the basis of the noiseless coding theorems, source entropies, and Huffman codes (and also on the basis of forecasting and experiments). Conversely, characterization theorems are given based on these properties. Some of the most important are Shannon's inequality, boundedness on an interval, subadditivity, additivity, branching, and expansibility. Also entropies of mixed probabilistic and nonprobabilistic character and convex $f$-divergences are mentioned, among others. Some unsolved problems are stated.

The converse to the noiseless coding theorem for average codeword lengths

$$\sum_{k=1}^{N} p_k n_k \geq - \sum_{k=1}^{N} p_k \log n_k = H(P),$$

(1)

where the $x_k$ are the message letters, their probabilities $p_k = p(x_k)$ add up to one, the $n_k$ are their codeword lengths in a uniquely decipherable code, $D$ is the number of symbols in the code alphabet, and $H$ is the Shannon entropy, is a consequence of the Shannon inequality

$$\sum p_k \log q_k \leq \sum p_k \log p_k$$

(2)

where $p_k > 0$, $q_k > 0$, and $\sum p_k = \sum q_k = 1$.

The direct, or positive, portion of the noiseless coding theorem states that for any $p_k > 0$ (or $p_k > 0$, $k = 1, 2, \cdots, N$) with $\sum p_k = 1$ there exists a uniquely decipherable code such that

$$\sum p_k n_k < H(P) + 1.$$  (3)

If $L$-tuples of messages are coded and $\mu_L$ denotes the average codeword length per message, then (3) can be improved to

$$\mu_L < \frac{H(P^L)}{L} + \frac{1}{L} < H(P) + \frac{1}{L}.$$  (4)

This is a consequence of the subadditivity property

$$H(PQ) < H(P) + H(Q)$$  (5)

where $P$ is as above,

$$PQ = \begin{pmatrix} y_1, & y_2, & \cdots, & y_M \\ p(y_1)p(y_2), & \cdots, & p(y_M) \end{pmatrix},$$

and

$$P^2 = \begin{pmatrix} x_1 \cap y_1, & \cdots, & x_1 \cap y_M \\ p(x_1 \cap y_1), & \cdots, & p(x_1 \cap y_M) \end{pmatrix} \cdots \begin{pmatrix} x_l \cap y_j, & \cdots, & x_l \cap y_M \\ p(x_l \cap y_j), & \cdots, & p(x_l \cap y_M) \end{pmatrix} \cdots$$

where $x_j \cap y_j$ means that the first message is $x_j$ and the second $y_j$, $x_l \cap x_l^j$ means that the first message is $x_l$ and the second $x_l$, and $P^n$ is interpreted similarly. Since the conditional entropy may be defined by

$$H(Q|P) = H(PQ) - H(P),$$

inequality (5) implies

$$H(Q|P) < H(Q).$$

More generally,

$$H(P_1, P_2, P_3, \cdots, P_L) = H(P_1) + H(P_2|P_1) + H(P_3|P_1, P_2) + \cdots + H(P_L|P_1, P_2, \cdots, P_{L-1})$$  (6)

and

$$H(Q|P_1, P_2, \cdots, P_j) \leq H(Q|P_1, P_2, \cdots, P_{j-1}).$$  (7)

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